

## Symmetry reduction in twisted noncommutative gravity with applications to cosmology and black holes

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# Symmetry reduction in twisted noncommutative gravity with applications to cosmology and black holes

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**ABSTRACT:** As a preparation for a mathematically consistent study of the physics of symmetric spacetimes in a noncommutative setting, we study symmetry reductions in deformed gravity. We focus on deformations that are given by a twist of a Lie algebra acting on the spacetime manifold. We derive conditions on those twists that allow a given symmetry reduction. A complete classification of admissible deformations is possible in a class of twists generated by commuting vector fields. As examples, we explicitly construct the families of vector fields that generate twists which are compatible with Friedmann-Robertson-Walker cosmologies and Schwarzschild black holes, respectively. We find nontrivial isotropic twists of FRW cosmologies and nontrivial twists that are compatible with all classical symmetries of black hole solutions.

**KEYWORDS:** Non-Commutative Geometry, Models of Quantum Gravity, Space-Time Symmetries.

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## 1. Introduction

The study of noncommutative geometry is an active topic in both theoretical physics and mathematics. From the mathematical perspective it is a generalization of classical (commutative) geometry. From the physics perspective it is suggested by the *Gedankenexperiment* of localizing events in spacetime with a Planck scale resolution [1]. In this *Gedankenexperiment*, a sharp localization induces an uncertainty in the spacetime coordinates, which can naturally be described by a noncommutative spacetime. Furthermore, noncommutative geometry and quantum gravity appear to be connected strongly and one can probably model “low energy” effects of quantum gravity theories using noncommutative geometry.

There have been many attempts to formulate scalar, gauge and gravity theories on noncommutative spacetime, in particular using the simplest example of a Moyal-Weyl spacetime having constant noncommutativity between space and time coordinates, see [2, 3] for reviews. Furthermore, this framework had been applied to phenomenological particle physics with [4, 5] and without Seiberg-Witten maps (see the review [6] and references therein), cosmology [7] and black hole physics (see the review [8] and references therein).

Our work is based on the approach outlined in [9–11], where a noncommutative gravity theory based on an arbitrary twist deformation is established. This approach has the advantages of being formulated using the symmetry principle of deformed diffeomorphisms,

being coordinate independent and applicable to nontrivial topologies. However, there is also the disadvantage that it does not match the Seiberg-Witten limit of string theory [12]. Nevertheless, string theory is not the only candidate for a fundamental theory of quantum gravity. Therefore, the investigation of deformed gravity remains interesting on its own terms and it could very well emerge from a fundamental theory of quantum gravity different from string theory.

The outline of this paper is as follows. In section 2 we review the basics of the formalism of twisted noncommutative differential geometry. For more details and the proofs we refer to the original paper [10] and the review [11]. We will work with a general twist and do not restrict ourselves to the Moyal-Weyl deformation.

In section 3 we will study symmetry reduction in theories based on twisted symmetries, such as the twisted diffeomorphisms in our theory of interest. The reason is that we aim to investigate which deformations of cosmological and black hole symmetries are possible. We will derive the conditions that the twist has to satisfy in order to be compatible with the reduced symmetry. In section 4 we restrict the twists to the class of Reshetikhin-Jambor-Sykora twists [13, 14], that are twists generated by commuting vector fields and are convenient for practical applications. Within this restricted class of twists we can classify more explicitly the possible deformations of Lie algebra symmetries acting on a manifold  $\mathcal{M}$ .

In section 5 and 6 we apply the formalism to cosmological symmetries as well as the black hole. We classify the possible Reshetikhin-Jambor-Sykora deformations of these models and obtain physically interesting ones. In section 7 we conclude and give an outlook to possible further investigations. In particular possible applications to phenomenological cosmology and black hole physics will be discussed.

## 2. Basics of twisted differential geometry and gravity

In order to establish notation, we will give a short summary of the framework of twisted differential geometry and gravity. More details can be found in [9–11].

There is a quite general procedure for constructing noncommutative spaces and their corresponding symmetries by using a twist. For this we require the following ingredients [11]:

1. a Lie algebra  $\mathfrak{g}$
2. an action of the Lie algebra on the space we want to deform
3. a twist element  $\mathcal{F}$ , constructed from the generators of the Lie algebra  $\mathfrak{g}$

By a twist element we denote an invertible element of  $U\mathfrak{g} \otimes U\mathfrak{g}$ , where  $U\mathfrak{g}$  is the universal enveloping algebra of  $\mathfrak{g}$ .  $\mathcal{F}$  has to fulfill some conditions, which will be specified later. The basic idea in the following is to combine any bilinear map with the inverse twist and therefore deform these maps. This leads to a mathematically consistent deformed theory covariant under the deformed transformations. We will show this now for the deformation of diffeomorphisms.

For our purpose we are interested in the Lie algebra of vector fields  $\Xi$  on a manifold  $\mathcal{M}$ . The transformations induced by  $\Xi$  can be seen as infinitesimal diffeomorphisms. A natural action of these transformations on the algebra of tensor fields  $\mathcal{T} := \bigoplus_{n,m} \otimes^n \Omega \otimes \otimes^m \Xi$  is given by the Lie derivative  $\mathcal{L}$ .  $\Omega$  denotes the space of one-forms.

In order to deform this Lie algebra, as well as its action on tensor fields and the tensor fields themselves, we first have to construct the enveloping algebra  $U\Xi$ . This is the associative tensor algebra generated by the elements of  $\Xi$  and the unit 1, modulo the left and right ideals generated by the elements  $[v, w] - vw + wv$ . This algebra can be seen as a Hopf algebra by using the following coproduct  $\Delta$ , antipode  $S$  and counit  $\epsilon$  defined on the generators  $u \in \Xi$  and 1 by:

$$\begin{aligned} \Delta(u) &= u \otimes 1 + 1 \otimes u, & \Delta(1) &= 1 \otimes 1, \\ \epsilon(u) &= 0, & \epsilon(1) &= 1, \\ S(u) &= -u, & S(1) &= 1. \end{aligned} \tag{2.1}$$

These definitions can be consistently carried over to the whole enveloping algebra demanding  $\Delta$  and  $\epsilon$  to be algebra homomorphisms and  $S$  to be an anti-homomorphism, i.e. for any two elements  $\eta, \xi \in U\Xi$ , we require

$$\Delta(\eta\xi) = \Delta(\eta)\Delta(\xi), \tag{2.2a}$$

$$\epsilon(\eta\xi) = \epsilon(\eta)\epsilon(\xi), \tag{2.2b}$$

$$S(\eta\xi) = S(\xi)S(\eta). \tag{2.2c}$$

The action of the enveloping algebra on the tensor fields can be defined by extending the Lie derivative

$$\mathcal{L}_{\eta\xi}(\tau) := \mathcal{L}_\eta(\mathcal{L}_\xi(\tau)), \quad \forall \eta, \xi \in U\Xi, \quad \tau \in \mathcal{T}. \tag{2.3}$$

This action is consistent with the Lie algebra properties, since  $\mathcal{L}_{[u,v]}(\tau) = \mathcal{L}_{uv}(\tau) - \mathcal{L}_{vu}(\tau)$  for all  $u, v \in \Xi$  by the properties of the Lie derivative.

The extension of the Lie algebra  $\Xi$  to the Hopf algebra  $(U\Xi, \cdot, \Delta, S, \epsilon)$ , where  $\cdot$  is the multiplication in  $U\Xi$ , can now be used in order to construct deformations of it. For the deformations we restrict ourselves to twist deformations, which is a wide class of possible deformations. The reason is that for twist deformations the construction of deformed differential geometry and gravity can be performed explicitly by only using properties of the twist, see [10]. Other deformations require further investigations.

In order to perform the deformation we require a twist element  $\mathcal{F} = f^\alpha \otimes f_\alpha \in U\Xi \otimes U\Xi$  (the sum over  $\alpha$  is understood) fulfilling the following conditions

$$\mathcal{F}_{12}(\Delta \otimes \text{id})\mathcal{F} = \mathcal{F}_{23}(\text{id} \otimes \Delta)\mathcal{F}, \tag{2.4a}$$

$$(\epsilon \otimes \text{id})\mathcal{F} = 1 = (\text{id} \otimes \epsilon)\mathcal{F}, \tag{2.4b}$$

$$\mathcal{F} = 1 \otimes 1 + \mathcal{O}(\lambda), \tag{2.4c}$$

where  $\mathcal{F}_{12} := \mathcal{F} \otimes 1$ ,  $\mathcal{F}_{23} := 1 \otimes \mathcal{F}$  and  $\lambda$  is the deformation parameter. The first condition will assure the associativity of the deformed products, the second will assure that deformed

multiplications with unit elements will be trivial and the third condition assures the existence of the undeformed classical limit  $\lambda \rightarrow 0$ . Furthermore, we can assume without loss of generality that  $f_\alpha$  (and also  $f^\alpha$ ) are linearly independent for all  $\alpha$ , what can be assured by combining linearly dependent  $f$ . Note that  $\mathcal{F}$  is regarded as formal power series in  $\lambda$ , such as the deformation itself. Strict (convergent) deformations will not be regarded here.

The simplest example is the twist on  $\mathbb{R}^n$  given by  $\mathcal{F}_\theta := \exp\left(-\frac{i\lambda}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu\right)$  with  $\theta^{\mu\nu} = \text{const.}$  and antisymmetric, leading to the Moyal-Weyl deformation, but there are also more complicated ones.

From a twist, one can construct the twisted triangular Hopf algebra  $(U\Xi_{\mathcal{F}}, \cdot, \Delta_{\mathcal{F}}, S_{\mathcal{F}}, \epsilon_{\mathcal{F}})$  with  $R$ -matrix  $R := \mathcal{F}_{21}\mathcal{F}^{-1} =: R^\alpha \otimes R_\alpha$ , inverse  $R^{-1} =: \bar{R}^\alpha \otimes \bar{R}_\alpha = R_{21}$  and

$$\Delta_{\mathcal{F}}(\xi) := \mathcal{F}\Delta(\xi)\mathcal{F}^{-1}, \quad \epsilon_{\mathcal{F}}(\xi) := \epsilon(\xi), \quad S_{\mathcal{F}}(\xi) := \chi S(\xi)\chi^{-1}, \quad (2.5)$$

where  $\chi := f^\alpha S(f_\alpha)$ ,  $\chi^{-1} := S(\bar{f}^\alpha)\bar{f}_\alpha$  and  $\bar{f}^\alpha \otimes \bar{f}_\alpha := \mathcal{F}^{-1}$ . Furthermore,  $\mathcal{F}_{21} := f_\alpha \otimes f^\alpha$  and  $R_{21} := R_\alpha \otimes R^\alpha$ . Again, we can assume without loss of generality that all summands of  $\mathcal{F}^{-1}$ ,  $R$  and  $R^{-1}$  are linearly independent.

However, as explained in [10], it is simpler to use the triangular  $\star$ -Hopf algebra  $\mathcal{H}_{\Xi}^\star = (U\Xi_\star, \star, \Delta_\star, S_\star, \epsilon_\star)$ , isomorphic to  $(U\Xi_{\mathcal{F}}, \cdot, \Delta_{\mathcal{F}}, S_{\mathcal{F}}, \epsilon_{\mathcal{F}})$ . The operations in this algebra on its generators  $u, v \in \Xi$  (note that this algebra has the same generators as the classical Hopf algebra) are defined by

$$u \star v := \bar{f}^\alpha(u)\bar{f}_\alpha(v), \quad (2.6a)$$

$$\Delta_\star(u) := u \otimes 1 + X_{\bar{R}^\alpha} \otimes \bar{R}_\alpha(u), \quad (2.6b)$$

$$\epsilon_\star(u) := \epsilon(u) = 0, \quad (2.6c)$$

$$S_\star^{-1}(u) := -\bar{R}^\alpha(u) \star X_{\bar{R}_\alpha}, \quad (2.6d)$$

where for all  $\xi \in U\Xi$  we define  $X_\xi := \bar{f}^\alpha \xi \chi S^{-1}(\bar{f}_\alpha)$ . The action of the twist on the elements of  $U\Xi$  is defined by extending the Lie derivative to the adjoint action [10]. Note that  $U\Xi = U\Xi_\star$  as vector spaces. The  $R$ -matrix is given by  $R_\star := X_{R^\alpha} \otimes X_{R_\alpha}$  and is triangular. The coproduct and antipode (2.6) is defined consistently on  $U\Xi_\star$  by using for all  $\xi, \eta \in U\Xi_\star$  the definitions

$$\Delta_\star(\xi \star \eta) := \Delta_\star(\xi) \star \Delta_\star(\eta), \quad S_\star(\xi \star \eta) := S_\star(\eta) \star S_\star(\xi). \quad (2.7)$$

The next step is to define the  $\star$ -Lie algebra of deformed infinitesimal diffeomorphisms. It has been shown [10] that for the twist deformation case the choice  $(\Xi_\star, [\cdot, \cdot]_\star)$ , where  $\Xi_\star = \Xi$  as vector spaces and

$$[u, v]_\star := [\bar{f}^\alpha(u), \bar{f}_\alpha(v)] \quad (2.8)$$

is a natural choice for a  $\star$ -Lie algebra. It fulfills all conditions which are necessary for a sensible  $\star$ -Lie algebra given by

1.  $\Xi_\star \subset U\Xi_\star$  is a linear space, which generates  $U\Xi_\star$

2.  $\Delta_\star(\Xi_\star) \subseteq \Xi_\star \otimes 1 + U\Xi_\star \otimes \Xi_\star$
3.  $[\Xi_\star, \Xi_\star]_\star \subseteq \Xi_\star$

The advantage of using the  $\star$ -Hopf algebra  $(U\Xi_\star, \star, \Delta_\star, S_\star, \epsilon_\star)$  instead of the  $\mathcal{F}$ -Hopf algebra  $(U\Xi_\mathcal{F}, \cdot, \Delta_\mathcal{F}, S_\mathcal{F}, \epsilon_\mathcal{F})$  is that the  $\star$ -Lie algebra of vector fields is isomorphic to  $\Xi$  as a vector space. For the  $\mathcal{F}$ -Hopf algebra this is not the case and the  $\mathcal{F}$ -Lie algebra consists in general of multidifferential operators.

The algebra of tensor fields  $\mathcal{T}$  is deformed by using the  $\star$ -tensor product [10]

$$\tau \otimes_\star \tau' := \bar{f}^\alpha(\tau) \otimes \bar{f}_\alpha(\tau'), \tag{2.9}$$

where as basic ingredients the deformed algebra of functions  $A_\star := (C^\infty(M), \star)$  as well as the  $A_\star$ -bimodules of vector fields  $\Xi_\star$  and one-forms  $\Omega_\star$  enter. We call  $\mathcal{T}_\star$  the deformed algebra of tensor fields. Note that  $\mathcal{T}_\star = \mathcal{T}$  as vector spaces.

The action of the deformed infinitesimal diffeomorphisms on  $\mathcal{T}_\star$  is defined by the  $\star$ -Lie derivative

$$\mathcal{L}_u^\star(\tau) := \mathcal{L}_{\bar{f}^\alpha(u)}(\bar{f}_\alpha(\tau)), \quad \forall \tau \in \mathcal{T}_\star, \quad u \in \Xi_\star, \tag{2.10}$$

which can be extended to all of  $U\Xi_\star$  by  $\mathcal{L}_{\xi_\star}^\star(\tau) := \mathcal{L}_\xi^\star(\mathcal{L}_\eta^\star(\tau))$ .

Furthermore, we define the  $\star$ -pairing  $\langle \cdot, \cdot \rangle_\star : \Xi_\star \otimes_{\mathbb{C}} \Omega_\star \rightarrow A_\star$  between vector fields and one-forms as

$$\langle v, \omega \rangle_\star := \langle \bar{f}^\alpha(v), \bar{f}_\alpha(\omega) \rangle, \quad \forall v \in \Xi_\star, \quad \omega \in \Omega_\star, \tag{2.11}$$

where  $\langle \cdot, \cdot \rangle$  is the undeformed pairing.

Based on the deformed symmetry principle one can define covariant derivatives, torsion and curvature. This leads to deformed Einstein equations, see [10], which we do not have to review here, since we do not use them in the following.

### 3. Symmetry reduction in twisted differential geometry

Assume that we have constructed a deformed gravity theory based on a twist  $\mathcal{F} \in U\Xi \otimes U\Xi$ . Like in Einstein gravity, the physical applications of this theory is strongly dependent on symmetry reduction. In this section we first define what we mean by symmetry reduction of a theory covariant under a Lie algebraic symmetry (e.g. infinitesimal diffeomorphisms) and then extend the principles to deformed symmetries and  $\star$ -Lie algebras.

In undeformed general relativity we often face the fact that the systems we want to describe have certain (approximate) symmetries. Here we restrict ourselves to Lie group symmetries. For example in cosmology one usually constrains oneself to fields invariant under certain symmetry groups  $G$ , like e.g. the euclidian group  $E_3$  for flat universes or the  $SO(4)$  group for universes with topology  $\mathbb{R} \times S_3$ , where the spatial hypersurfaces are 3-spheres. For a non rotating black hole one usually demands the metric to be stationary and spherically symmetric. Practically, one uses the corresponding Lie algebra  $\mathfrak{g}$  of the symmetry group  $G$ , represents it faithfully on the Lie algebra of vector fields  $\Xi$  on the

manifold  $\mathcal{M}$  and demands the fields  $\tau \in \mathcal{T}$ , which occur in the theory, to be invariant under these transformations, i.e. we demand

$$\mathcal{L}_v(\tau) = 0, \quad \forall v \in \mathfrak{g}. \quad (3.1)$$

Since the Lie algebra  $\mathfrak{g}$  is a linear space we can choose a basis  $\{t_i : i = 1, \dots, \dim(\mathfrak{g})\}$  and can equivalently demand

$$\mathcal{L}_{t_i}(\tau) = 0, \quad \forall i = 1, 2, \dots, \dim(\mathfrak{g}). \quad (3.2)$$

The Lie bracket of the generators has to fulfill

$$[t_i, t_j] = f_{ij}{}^k t_k, \quad (3.3)$$

where  $f_{ij}{}^k$  are the structure constants.

One can easily show that if we combine two invariant tensors with the tensor product, the resulting tensor is invariant too because of the trivial coproduct

$$\mathcal{L}_{t_i}(\tau \otimes \tau') = \mathcal{L}_{t_i}(\tau) \otimes \tau' + \tau \otimes \mathcal{L}_{t_i}(\tau'). \quad (3.4)$$

The same holds true for pairings  $\langle v, \omega \rangle$  of invariant objects  $v \in \Xi$  and  $\omega \in \Omega$ .

Furthermore, if a tensor is invariant under infinitesimal transformations, it is also invariant under (at least small) finite transformations, since they are given by exponentiating the generators. The exponentiated generators are part of the enveloping algebra, i.e.  $\exp(\alpha^i t_i) \in U\mathfrak{g}$ , where  $\alpha^i$  are parameters. For large finite transformations the topology of the Lie group can play a role, such that the group elements may not simply be given by exponentiating the generators. In the following we will focus only on small finite transformations in order to avoid topological effects.

We now generalize this to the case of  $\star$ -Hopf algebras and their corresponding  $\star$ -Lie algebras. Our plan is as follows: we start with a suitable definition of a  $\star$ -Lie subalgebra constructed from the Lie algebra  $(\mathfrak{g}, [ , ])$ . This definition is guided by conditions, which allow for deformed symmetry reduction using infinitesimal transformations. Then we complete this  $\star$ -Lie subalgebra in several steps to a  $\star$ -enveloping subalgebra, a  $\star$ -Hopf subalgebra and a triangular  $\star$ -Hopf subalgebra. We will always be careful that the dimension of the  $\star$ -Lie subalgebra remains the same as the dimension of the corresponding classical Lie algebra. At each step we obtain several restrictions between the twist and  $(\mathfrak{g}, [ , ])$ .

We start by taking the generators  $\{t_i\}$  of  $\mathfrak{g} \subseteq \Xi$  and representing their deformations in the  $\star$ -Lie algebra  $(\Xi_\star, [ , ]_\star)$  as

$$t_i^\star = t_i + \sum_{n=1}^{\infty} \lambda^n t_i^{(n)}, \quad (3.5)$$

where  $\lambda$  is the deformation parameter and  $t_i^{(n)} \in \Xi_\star$ .

The span of these deformed generators, together with the  $\star$ -Lie bracket, should form a  $\star$ -Lie subalgebra  $(\mathfrak{g}_\star, [ , ]_\star) := (\text{span}(t_i^\star), [ , ]_\star)$ . Therefore  $(\mathfrak{g}_\star, [ , ]_\star)$  has to obey certain



conditions. Natural conditions are

$$[\mathfrak{g}_\star, \mathfrak{g}_\star]_\star \subseteq \mathfrak{g}_\star, \quad \text{i.e. } [t_i^\star, t_j^\star]_\star = f_{ij}^{\star k} t_k^\star \text{ with } f_{ij}^{\star k} = f_{ij}^k + \mathcal{O}(\lambda) \quad (3.6a)$$

$$\Delta_\star(\mathfrak{g}_\star) \subseteq \mathfrak{g}_\star \otimes 1 + U\Xi_\star \otimes \mathfrak{g}_\star, \quad \text{which is equivalent to } \bar{R}_\alpha(\mathfrak{g}_\star) \subseteq \mathfrak{g}_\star \quad \forall \alpha \quad (3.6b)$$

The first condition is a basic feature of a  $\star$ -Lie algebra. The second condition implies that if we have two  $\mathfrak{g}_\star$  invariant tensors  $\tau, \tau' \in \mathcal{T}_\star$ , the  $\star$ -tensor product of them is invariant as well

$$\mathcal{L}_{t_i^\star}^\star(\tau \otimes_\star \tau') = \mathcal{L}_{t_i^\star}^\star(\tau) \otimes_\star \tau' + \bar{R}^\alpha(\tau) \otimes_\star \mathcal{L}_{\bar{R}_\alpha(t_i^\star)}^\star(\tau') = 0, \quad (3.7)$$

since  $\bar{R}_\alpha(t_i^\star) \in \mathfrak{g}_\star$ . The  $\star$ -pairings  $\langle v, \omega \rangle_\star$  of two invariant objects  $v \in \Xi_\star$  and  $\omega \in \Omega_\star$  are also invariant under the  $\star$ -action of  $\mathfrak{g}_\star$ . These are important features if one wants to combine invariant objects to e.g. an invariant action. Furthermore, the conditions are sufficient such that the following consistency relation is fulfilled for any invariant tensor  $\tau \in \mathcal{T}_\star$

$$0 = f_{ij}^{\star k} \mathcal{L}_{t_k^\star}^\star(\tau) = \mathcal{L}_{[t_i^\star, t_j^\star]}^\star(\tau) = \mathcal{L}_{t_i^\star}^\star(\mathcal{L}_{t_j^\star}^\star(\tau)) - \mathcal{L}_{\bar{R}_\alpha(t_j^\star)}^\star(\mathcal{L}_{\bar{R}_\alpha(t_i^\star)}^\star(\tau)), \quad (3.8)$$

since  $\bar{R}_\alpha(t_i^\star) \in \mathfrak{g}_\star$ .

Hence by demanding the two conditions (3.6) for the  $\star$ -Lie subalgebra  $(\text{span}(t_i^\star), [ , ]_\star)$  we can consistently perform symmetry reduction by using deformed *infinitesimal* transformations. In the classical limit  $\lambda \rightarrow 0$  we obtain the classical Lie algebra  $(\mathfrak{g}_\star, [ , ]_\star) \xrightarrow{\lambda \rightarrow 0} (\mathfrak{g}, [ , ])$ .

Next, we consider the extension of the  $\star$ -Lie subalgebra  $(\mathfrak{g}_\star, [ , ]_\star) \subseteq (\Xi_\star, [ , ]_\star)$  to the triangular  $\star$ -Hopf subalgebra  $\mathcal{H}_\mathfrak{g}^\star = (U\mathfrak{g}_\star, \star, \Delta_\star, S_\star, \epsilon_\star) \subseteq \mathcal{H}_\Xi^\star$ . This can be seen as extending the infinitesimal transformations to a quantum group. We will divide this path into several steps, where in every step we have to demand additional restrictions on the twist.

Firstly, we construct the  $\star$ -tensor algebra generated by the elements of  $\mathfrak{g}_\star$  and 1. We take this tensor algebra modulo the left and right ideals generated by the elements  $[u, v]_\star - u \star v + \bar{R}^\alpha(v) \star \bar{R}_\alpha(u)$ . It is necessary that these elements are part of  $U\mathfrak{g}_\star$ , i.e. we require

$$\bar{R}^\alpha(\mathfrak{g}_\star) \star \bar{R}_\alpha(\mathfrak{g}_\star) \subseteq U\mathfrak{g}_\star. \quad (3.9)$$

This leads to the algebra  $(U\mathfrak{g}_\star, \star)$ , which is a subalgebra of  $(U\Xi_\star, \star)$ .

Secondly, we extend this subalgebra to a  $\star$ -Hopf subalgebra. Therefore we additionally have to require that

$$\Delta_\star(U\mathfrak{g}_\star) \subseteq U\mathfrak{g}_\star \otimes U\mathfrak{g}_\star, \quad (3.10a)$$

$$S_\star(U\mathfrak{g}_\star) \subseteq U\mathfrak{g}_\star. \quad (3.10b)$$

Note that we do not demand that  $S_\star^{-1}$  (defined on  $U\Xi_\star$ ) closes in  $U\mathfrak{g}_\star$ , since this is in general not the case for a nonquasitriangular Hopf algebra and we do not want to demand quasitriangularity at this stage. Then the  $\star$ -Hopf algebra  $\mathcal{H}_\mathfrak{g}^\star$  is a Hopf subalgebra of  $\mathcal{H}_\Xi^\star$ .

Thirdly, we additionally demand that there exists an  $R$ -matrix  $R_\star \in U\mathfrak{g}_\star \otimes U\mathfrak{g}_\star$ . It is natural to take the  $R$ -matrix of the triangular  $\star$ -Hopf algebra  $\mathcal{H}_\Xi^\star$  defined by  $R_\star := X_{R^\alpha} \otimes X_{R_\alpha}$ . This leads to the restrictions

$$X_{R^\alpha}, X_{R_\alpha} \in U\mathfrak{g}_\star, \quad \forall \alpha. \quad (3.11)$$

Since  $R_\star$  is triangular, i.e.  $R_\star^{-1} = \bar{R}_\star^\alpha \otimes \bar{R}_{\star\alpha} = R_{\star 21} = R_{\star\alpha} \otimes R_\star^\alpha$ , we also have  $X_{\bar{R}^\alpha}, X_{\bar{R}_\alpha} \in U\mathfrak{g}_\star, \forall_\alpha$ . If these conditions are fulfilled,  $\mathcal{H}_\mathfrak{g}^\star$  is a triangular  $\star$ -Hopf subalgebra of  $\mathcal{H}_\Xi^\star$  with the same  $R$ -matrix.

As we have seen, extending the  $\star$ -Lie subalgebra to a (triangular)  $\star$ -Hopf subalgebra gives severe restrictions on the possible deformations, more than just working with the deformed infinitesimal transformations given by a  $\star$ -Lie subalgebra or the finite transformations given by the  $\star$ -enveloping subalgebra  $(U\mathfrak{g}_\star, \star)$ . Now the question arises if we actually require the deformed finite transformations to form a (triangular)  $\star$ -Hopf algebra in order to use them for a sensible symmetry reduction. Because  $(U\mathfrak{g}_\star, \star)$  describes deformed finite transformations and we have the relation

$$\mathcal{L}_{U\mathfrak{g}_\star \setminus \{1\}}^\star(\tau) = \{0\} \Leftrightarrow \mathcal{L}_{\mathfrak{g}_\star}^\star(\tau) = \{0\}, \tag{3.12}$$

we can consistently demand tensors to be invariant under  $(U\mathfrak{g}_\star, \star)$ , since we require tensors to be invariant under  $(\mathfrak{g}_\star, [\ , \ ]_\star)$ . Therefore, a well defined  $(U\mathfrak{g}_\star, \star)$  leads to a structure sufficient for symmetry reduction. The equivalence (3.12) can be shown by using linearity of the  $\star$ -Lie derivative and the property  $\mathcal{L}_{\xi^\star\eta}^\star(\tau) = \mathcal{L}_\xi^\star(\mathcal{L}_\eta^\star(\tau))$ .

In order to better understand the different restrictions necessary for constructing the  $\star$ -Lie subalgebra  $(\mathfrak{g}_\star, [\ , \ ]_\star)$ , the  $\star$ -enveloping subalgebra and the (triangular)  $\star$ -Hopf subalgebra  $(U\mathfrak{g}_\star, \star, \Delta_\star, S_\star, \epsilon_\star)$ , we restrict ourselves in the following sections to the class of Reshetikhin-Jambor-Sykora twists [13, 14]. This is a suitable nontrivial generalization of the Moyal-Weyl product, also containing e.g.  $\kappa$  and  $q$  deformations when applied to Poincaré symmetry.

#### 4. The case of Reshetikhin-Jambor-Sykora twists

Let  $\{V_a \in \Xi\}$  be an arbitrary set of mutually commuting vector fields, i.e.  $[V_a, V_b] = 0, \forall_{a,b}$ , on an  $n$  dimensional manifold  $\mathcal{M}$ . Then the object

$$\mathcal{F}_V := \exp\left(-\frac{i\lambda}{2}\theta^{ab}V_a \otimes V_b\right) \in U\Xi \otimes U\Xi \tag{4.1}$$

is a twist element, if  $\theta$  is constant and antisymmetric [10, 13, 14]. We call (4.1) a Reshetikhin-Jambor-Sykora twist. Note that this twist is not restricted to the topology  $\mathbb{R}^n$  for the manifold  $\mathcal{M}$ .

Furthermore, we can restrict ourselves to  $\theta$  with maximal rank and an even number of vector fields  $V_a$ , since we can lower the rank of the Poisson structure afterwards by choosing some of the  $V_a$  to be zero. We can therefore without loss of generality use the standard form

$$\theta = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ -1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & -1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{4.2}$$

by applying a suitable  $GL(n)$  transformation on the  $V_a$ .

This twist element is easy to apply and in particular we have for the inverse and the  $R$ -matrix

$$\mathcal{F}_V^{-1} = \exp\left(\frac{i\lambda}{2}\theta^{ab}V_a \otimes V_b\right), \quad R = \mathcal{F}_{V,21}\mathcal{F}_V^{-1} = \mathcal{F}_V^{-2} = \exp(i\lambda\theta^{ab}V_a \otimes V_b). \quad (4.3)$$

Now let  $(\mathfrak{g}, [\ , \ ]) \subseteq (\Xi, [\ , \ ])$  be the Lie algebra of the symmetry we want to deform. We choose a basis of this Lie algebra  $\{t_i : i = 1, \dots, \dim(\mathfrak{g})\}$  with  $[t_i, t_j] = f_{ij}{}^k t_k$ .

Next, we discuss the symmetry reduction based on the  $\star$ -Lie subalgebra, as explained in section 3. Therefore we make the ansatz (3.5) for the generators  $t_i^*$ . Furthermore, we evaluate the two conditions (3.6) the  $t_i^*$  have to satisfy. We start with the coproduct condition (3.6b), which is equivalent to  $\bar{R}_\alpha(t_i^*) \in \text{span}(t_i^*)$ ,  $\forall \alpha$ , where  $\alpha$  is a multi index. Using the explicit form of the inverse  $R$ -matrix (4.3) we arrive at the conditions

$$[V_{a_1}, \dots [V_{a_n}, t_i^*] \dots] = \mathcal{N}_{a_1 \dots a_n i}^{\star j} t_j^*, \quad (4.4)$$

where  $\mathcal{N}_{a_1 \dots a_n i}^{\star j} := \mathcal{N}_{a_1 \dots a_n i}^j + \sum_{k=1}^{\infty} \lambda^k \mathcal{N}_{a_1 \dots a_n i}^{(k) j}$  are constants.

The only independent condition in (4.4) is given by

$$[V_a, t_i^*] = \mathcal{N}_{ai}^{\star j} t_j^*, \quad (4.5)$$

since it implies all the other ones by linearity. In particular, the zeroth order in  $\lambda$  of (4.5) yields

$$[V_a, t_i] = \mathcal{N}_{ai}^j t_j. \quad (4.6)$$

This leads to the following

**Proposition 1.** *Let  $(\mathfrak{g}, [\ , \ ]) \subseteq (\Xi, [\ , \ ])$  be a classical Lie algebra and  $(\Xi_\star, [\ , \ ]_\star)$  the  $\star$ -Lie algebra of vector fields deformed by a Reshetikhin-Jambor-Sykora twist, constructed with vector fields  $V_a$ . Then for a symmetry reduction respecting the minimal axioms (3.6), it is necessary that the following Lie bracket relations hold true*

$$[V_a, \mathfrak{g}] \subseteq \mathfrak{g}, \forall a. \quad (4.7)$$

In other words,  $(\text{span}(t_i, V_a), [\ , \ ]) \subseteq (\Xi, [\ , \ ])$  forms a Lie algebra with ideal  $\mathfrak{g}$ . Here  $t_i$  are the generators of  $\mathfrak{g}$ .

Note that this gives conditions relating the classical Lie algebra  $(\mathfrak{g}, [\ , \ ])$  with the twist.

Next, we evaluate the  $\star$ -Lie bracket condition (3.6a). Using the explicit form of the inverse twist (4.3) and (4.5) we obtain

$$\bar{f}_{\alpha(n)}(t_i^*) = [V_{a_1}, \dots [V_{a_n}, t_i^*] \dots] = (\mathcal{N}_{a_n}^* \dots \mathcal{N}_{a_1}^*)^j t_j^* =: \mathcal{N}_{\alpha(n)i}^{\star j} t_j^*, \quad (4.8a)$$

$$\bar{f}^{\alpha(n)}(t_i^*) = \Theta^{\beta(n)\alpha(n)} \bar{f}_{\beta(n)}(t_i^*), \quad (4.8b)$$

$$\Theta^{\beta(n)\alpha(n)} := \frac{1}{n!} \left(\frac{i\lambda}{2}\right)^n \theta^{b_1 a_1} \dots \theta^{b_n a_n}, \quad (4.8c)$$

where  $\alpha_{(n)}, \beta_{(n)}$  are multi indices. This leads to

$$[t_i^*, t_j^*]_\star = \Theta^{\beta_{(n)}\alpha_{(n)}} \mathcal{N}_{\beta_{(n)}i}^{\star k} \mathcal{N}_{\alpha_{(n)}j}^{\star l} [t_k^*, t_l^*]. \quad (4.9)$$

Note that in particular for the choice  $t_i^* = t_i, \forall_i$ , the  $\star$ -Lie subalgebra closes with structure constants

$$[t_i, t_j]_\star = \Theta^{\beta_{(n)}\alpha_{(n)}} \mathcal{N}_{\beta_{(n)}i}^k \mathcal{N}_{\alpha_{(n)}j}^l [t_k, t_l] = \Theta^{\beta_{(n)}\alpha_{(n)}} \mathcal{N}_{\beta_{(n)}i}^k \mathcal{N}_{\alpha_{(n)}j}^l f_{kl}^m t_m =: f_{ij}^{\star m} t_m, \quad (4.10)$$

where we have used the  $\mathcal{N}$  defined in (4.6). This leads to the following

**Proposition 2.** *Let  $[V_a, \mathfrak{g}_\star] \subseteq \mathfrak{g}_\star, \forall_a$ . Then we can always construct a  $\star$ -Lie subalgebra  $(\mathfrak{g}_\star, [ , ]_\star) \subseteq (\Xi_\star, [ , ]_\star)$  by choosing the generators as  $t_i^* = t_i$  for all  $i$ . With this we have  $\mathfrak{g}_\star = \mathfrak{g}$  as vector spaces and the structure constants are deformed as*

$$f_{ij}^{\star m} = \Theta^{\beta_{(n)}\alpha_{(n)}} \mathcal{N}_{\beta_{(n)}i}^k \mathcal{N}_{\alpha_{(n)}j}^l f_{kl}^m. \quad (4.11)$$

Since the condition (3.6b) together with the requirement  $t_i^* = t_i$ , for all  $i$ , automatically fulfills (3.6a), we choose  $t_i^* = t_i$ , for all  $i$ , as a canonical embedding. In general, other possible embeddings require further constructions to fulfill condition (3.6a) and are therefore less natural. We will discuss possible differences between this and other embeddings later on, when we construct the  $\star$ -Hopf subalgebra and the  $\star$ -Lie derivative action on  $\star$ -tensor fields.

In addition, we obtain that the necessary condition (3.9) for extending  $\mathfrak{g}_\star$  to the  $\star$ -enveloping subalgebra  $(U\mathfrak{g}_\star, \star) \subseteq (U\Xi_\star, \star)$  is automatically fulfilled, since we have  $\bar{R}_{\alpha_{(n)}}(\mathfrak{g}_\star) \subseteq \mathfrak{g}_\star$  for all  $\alpha_{(n)}$  and additionally

$$\bar{R}^{\alpha_{(n)}}(\mathfrak{g}_\star) = (-2)^n \Theta^{\beta_{(n)}\alpha_{(n)}} \bar{R}_{\beta_{(n)}}(\mathfrak{g}_\star) \subseteq \mathfrak{g}_\star, \quad \forall \alpha_{(n)}. \quad (4.12)$$

Next, we evaluate the conditions (3.10), which have to be fulfilled in order to construct the  $\star$ -Hopf subalgebra  $\mathcal{H}_\mathfrak{g}^\star \subseteq \mathcal{H}_\Xi^\star$ . For the particular choice of the twist (4.1) we obtain the following

**Proposition 3.** *Let  $(U\mathfrak{g}_\star, \star) \subseteq (U\Xi_\star, \star)$  be a  $\star$ -enveloping subalgebra and let the deformation parameter  $\lambda \neq 0$ . Then in order to extend  $(U\mathfrak{g}_\star, \star)$  to the  $\star$ -Hopf subalgebra  $\mathcal{H}_\mathfrak{g}^\star = (U\mathfrak{g}_\star, \star, \Delta_\star, S_\star, \epsilon_\star) \subseteq \mathcal{H}_\Xi^\star$  the condition*

$$V_{a_1} \in \mathfrak{g}_\star, \quad \text{if } [V_{a_2}, \mathfrak{g}_\star] \neq \{0\} \quad (4.13)$$

has to hold true for all pairs of indices  $(a_1, a_2)$  connected by the antisymmetric matrix  $\theta$  (4.2), i.e.  $(a_1, a_2) \in \{(1, 2), (2, 1), (3, 4), (4, 3), \dots\}$ .

Note that these conditions depend on the embedding  $t_i^* = t_i(t_j)$ . The proof of this proposition is shown in the appendix A.

Finally, if we demand  $\mathcal{H}_\mathfrak{g}^\star$  to be a triangular  $\star$ -Hopf algebra (3.11) we obtain the stringent condition

$$V_a \in \mathfrak{g}_\star, \quad \forall_a. \quad (4.14)$$

This can be shown by using  $X_{R_\alpha} = R_\alpha$  and  $V_a \star V_b = V_a V_b$ , which holds true for the class of Reshetikhin-Jambor-Sykora twists.

As we have seen above, there are much stronger restrictions on the Lie algebra  $(\mathfrak{g}, [ , ])$  and the twist, if we want to extend the deformed infinitesimal transformations  $(\mathfrak{g}_\star, [ , ]_\star)$  to the (triangular)  $\star$ -Hopf subalgebra  $\mathcal{H}_\mathfrak{g}^\star$ . In particular this extension restricts the  $V_a$  themselves, while for infinitesimal transformations and the finite transformations  $(U_{\mathfrak{g}_\star}, \star)$  only the images of  $V_a$  acting on  $\mathfrak{g}_\star$  are important.

Next, we study the  $\star$ -action of the  $\star$ -Lie and Hopf algebra on the deformed tensor fields. The  $\star$ -action of the generators  $t_i^\star$  on  $\tau \in \mathcal{T}_\star$  is defined by (2.10) and simplifies to

$$\mathcal{L}_{t_i^\star}^\star(\tau) = \Theta^{\alpha(n)\beta(n)} \mathcal{N}_{\alpha(n)i}^{\star j} \mathcal{L}_{t_j^\star}(\bar{f}_{\beta(n)}(\tau)) . \tag{4.15}$$

For invariant tensors, the  $\star$ -Lie derivative has to vanish to all orders in  $\lambda$ , since we work with formal power series. If we now for explicitness take the natural choice  $t_i^\star = t_i$  we obtain the following

**Proposition 4.** *Let  $[V_a, \mathfrak{g}_\star] \subseteq \mathfrak{g}_\star, \forall_a$  and  $t_i^\star = t_i, \forall_i$ . Then a tensor  $\tau \in \mathcal{T}_\star$  is  $\star$ -invariant under  $(\mathfrak{g}_\star, [ , ]_\star)$ , if and only if it is invariant under the undeformed action of  $(\mathfrak{g}, [ , ])$ , i.e.*

$$\mathcal{L}_{\mathfrak{g}_\star}^\star(\tau) = \{0\} \Leftrightarrow \mathcal{L}_\mathfrak{g}(\tau) = \{0\} . \tag{4.16}$$

*Proof.* For the proof we make the ansatz  $\tau = \sum_{n=0}^\infty \lambda^n \tau_n$  and investigate  $\mathcal{L}_{t_i^\star}^\star(\tau)$  order by order in  $\lambda$ , since we work with formal power series. By using (4.6) to reorder the Lie derivatives such that  $t_i$  is moved to the right, it can be shown recursively in powers of  $\lambda$  that the proposition holds true.  $\square$

Note that for  $t_i^\star \neq t_i$  this does not necessarily hold true. We can not make statements for this case, since we would require a general solution of (4.9), which we do not have yet. But we mention again that we consider choosing  $t_i^\star$  different from  $t_i$  quite unnatural.

This proposition translates to the case of finite symmetry transformations with  $t_i^\star = t_i$  because of the properties of the  $\star$ -Lie derivative.

The framework developed in this section will now be applied to cosmology and black holes in order to give some specific examples and discuss possible physical implications.

## 5. Application to cosmology

In this section we will investigate models with symmetry group  $E_3$  in four spacetime dimensions with topology  $\mathbb{R}^4$ . These are flat Friedmann-Robertson-Walker (FRW) universes. The undeformed Lie algebra of this group is generated by the “momenta”  $p_i$  and “angular momenta”  $L_i, i \in \{1, 2, 3\}$ , which we can represent in the Lie algebra of vector fields as

$$p_i = \partial_i, \quad L_i = \epsilon_{ijk} x^j \partial_k, \tag{5.1}$$

where  $\epsilon_{ijk}$  is the Levi-Civita symbol.

The undeformed Lie bracket relations are

$$[p_i, p_j] = 0, \quad [p_i, L_j] = -\epsilon_{ijk} p_k, \quad [L_i, L_j] = -\epsilon_{ijk} L_k. \quad (5.2)$$

We will work with the natural embedding  $t_i^* = t_i$ , and therefore the  $\star$ -Lie subalgebra is given by  $\mathfrak{g}_\star = \mathfrak{e}_{3\star} = \mathfrak{e}_3 = \text{span}(p_i, L_i)$ .

We can now explicitly evaluate the condition each twist vector field  $V_a$  has to satisfy given by  $[V_a, \mathfrak{e}_3] \subseteq \mathfrak{e}_3$  (cf. proposition 1). Since the generators are at most linear in the spatial coordinates,  $V_a$  can be at most quadratic in order to fulfill this condition. If we make a quadratic ansatz with time dependent coefficients we obtain that each  $V_a$  has to be of the form

$$V_a = V_a^0(t) \partial_t + c_a^i \partial_i + d_a^i L_i + f_a x^i \partial_i, \quad (5.3)$$

where  $c_a^i, d_a^i, f_a \in \mathbb{R}$  and  $V_a^0(t) \in C^\infty(\mathbb{R})$  in order to obtain hermitian deformations. If all  $V_a$  have the form (5.3), the  $\star$ -Lie algebra closes (cf. proposition 2).

Next, we have to find conditions such that the  $V_a$  are mutually commuting. A brief calculation shows that the following conditions have to be fulfilled:

$$d_a^i d_b^j \epsilon_{ijk} = 0, \quad \forall_k, \quad (5.4a)$$

$$c_a^i d_b^j \epsilon_{ijk} - c_b^i d_a^j \epsilon_{ijk} + f_a c_b^k - f_b c_a^k = 0, \quad \forall_k, \quad (5.4b)$$

$$[V_a^0(t) \partial_t, V_b^0(t) \partial_t] = 0. \quad (5.4c)$$

As a first step, we will now work out all possible deformations of  $\mathfrak{e}_3$  when twisted with two commuting vector fields. We will classify the possible solutions. Therefore we divide the solutions into classes depending on the value of  $d_a^i$  and  $f_a$ . We use as notation for our cosmologies  $\mathfrak{C}_{AB}$ , where  $A \in \{1, 2, 3\}$  and  $B \in \{1, 2\}$ , which will become clear later on, when we sum up the results in table 1.

Type  $\mathfrak{C}_{11}$  is defined to be vector fields with  $d_1^i = d_2^i = 0$  and  $f_1 = f_2 = 0$ , i.e.

$$V_{1(\mathfrak{C}_{11})} = V_1^0(t) \partial_t + c_1^i \partial_i, \quad V_{2(\mathfrak{C}_{11})} = V_2^0(t) \partial_t + c_2^i \partial_i. \quad (5.5)$$

These vector fields fulfill the first two conditions (5.4a) and (5.4b). The solutions of the third condition (5.4c) will be discussed later, since this classification we perform now does not depend on it.

Type  $\mathfrak{C}_{21}$  is defined to be vector fields with  $d_1^i = d_2^i = 0$ ,  $f_1 \neq 0$  and  $f_2 = 0$ . The first condition (5.4a) is trivially fulfilled and the second (5.4b) is fulfilled, if and only if  $c_2^i = 0$ ,  $\forall_i$ , i.e. type  $\mathfrak{C}_{21}$  is given by the vector fields

$$\tilde{V}_{1(\mathfrak{C}_{21})} = V_1^0(t) \partial_t + c_1^i \partial_i + f_1 x^i \partial_i, \quad \tilde{V}_{2(\mathfrak{C}_{21})} = V_2^0(t) \partial_t. \quad (5.6)$$

These vector fields can be simplified to

$$V_{1(\mathfrak{C}_{21})} = c_1^i \partial_i + f_1 x^i \partial_i, \quad V_{2(\mathfrak{C}_{21})} = V_2^0(t) \partial_t, \quad (5.7)$$

since both lead to the same twist (4.1).

$\mathfrak{C}_{AB}$	$\mathbf{d}_1 = \mathbf{d}_2 = 0$	$\mathbf{d}_1 \neq 0, \mathbf{d}_2 = 0$
$f_1 = 0,$ $f_2 = 0$	$V_1 = V_1^0(t)\partial_t + c_1^i\partial_i$ $V_2 = V_2^0(t)\partial_t + c_2^i\partial_i$	$V_1 = V_1^0(t)\partial_t + c_1^i\partial_i + d_1^i L_i$ $V_2 = V_2^0(t)\partial_t + \kappa d_1^i\partial_i$
$f_1 \neq 0,$ $f_2 = 0$	$V_1 = c_1^i\partial_i + f_1 x^i\partial_i$ $V_2 = V_2^0(t)\partial_t$	$V_1 = c_1^i\partial_i + d_1^i L_i + f_1 x^i\partial_i$ $V_2 = V_2^0(t)\partial_t$
$f_1 = 0,$ $f_2 \neq 0$	$V_1 = V_1^0(t)\partial_t$ $V_2 = c_2^i\partial_i + f_2 x^i\partial_i$	$V_1 = V_1^0(t)\partial_t + \frac{1}{f_2} d_1^j c_2^k \epsilon_{jki} \partial_i + d_1^i L_i$ $V_2 = V_2^0(t)\partial_t + c_2^i\partial_i + f_2 x^i\partial_i$

**Table 1:** Two vector field deformations of the cosmological symmetry group  $E_3$ .

Solutions with  $d_1^i = d_2^i = 0$ ,  $f_1 \neq 0$  and  $f_2 \neq 0$  lie in type  $\mathfrak{C}_{21}$ , since we can perform the twist conserving map  $V_2 \rightarrow V_2 - \frac{f_2}{f_1} V_1$ , which transforms  $f_2$  to zero. Furthermore  $\mathfrak{C}_{31}$  is defined by  $d_1^i = d_2^i = 0$ ,  $f_1 = 0$  and  $f_2 \neq 0$  and is equivalent to  $\mathfrak{C}_{21}$  by interchanging the labels of the vector fields.

Next, we go on to solutions with without loss of generality  $\mathbf{d}_1 \neq 0$  and  $\mathbf{d}_2 = 0$  ( $\mathbf{d}$  denotes the vector). Note that this class contains also the class with  $\mathbf{d}_1 \neq 0$  and  $\mathbf{d}_2 \neq 0$ . To see this, we use the first condition (5.4a) and obtain that  $\mathbf{d}_1$  and  $\mathbf{d}_2$  have to be parallel, i.e.  $d_2^i = \kappa d_1^i$ . Then we can transform  $\mathbf{d}_2$  to zero by using the twist conserving map  $V_2 \rightarrow V_2 - \kappa V_1$ .

Type  $\mathfrak{C}_{12}$  is defined to be vector fields with  $\mathbf{d}_1 \neq 0$ ,  $\mathbf{d}_2 = 0$  and  $f_1 = f_2 = 0$ . The first condition (5.4a) is trivially fulfilled, while the second condition (5.4b) requires that  $\mathbf{c}_2$  is parallel to  $\mathbf{d}_1$ , i.e. we obtain

$$V_{1(\mathfrak{C}_{12})} = V_1^0(t)\partial_t + c_1^i\partial_i + d_1^i L_i, \quad V_{2(\mathfrak{C}_{12})} = V_2^0(t)\partial_t + \kappa d_1^i\partial_i, \quad (5.8)$$

where  $\kappa \in \mathbb{R}$  is a constant.

Type  $\mathfrak{C}_{22}$  is defined to be vector fields with  $\mathbf{d}_1 \neq 0$ ,  $\mathbf{d}_2 = 0$ ,  $f_1 \neq 0$  and  $f_2 = 0$ . Solving the second condition (5.4b) (therefore we have to use that the vectors are real!) we obtain

$$V_{1(\mathfrak{C}_{22})} = c_1^i\partial_i + d_1^i L_i + f_1 x^i\partial_i, \quad V_{2(\mathfrak{C}_{22})} = V_2^0(t)\partial_t, \quad (5.9)$$

where we could set without loss of generality  $V_1^0(t)$  to zero, as in type  $\mathfrak{C}_{21}$ . Note that  $\mathfrak{C}_{21}$  is contained in  $\mathfrak{C}_{22}$  by violating the condition  $\mathbf{d}_1 \neq 0$ .

Finally, we come to the last class, type  $\mathfrak{C}_{32}$ , defined by  $\mathbf{d}_1 \neq 0$ ,  $\mathbf{d}_2 = 0$ ,  $f_1 = 0$  and  $f_2 \neq 0$ . This class contains also the case  $\mathbf{d}_1 \neq 0$ ,  $\mathbf{d}_2 = 0$ ,  $f_1 \neq 0$  and  $f_2 \neq 0$  by using the twist conserving map  $V_1 \rightarrow V_1 - \frac{f_1}{f_2} V_2$ . The vector fields are given by

$$V_{1(\mathfrak{C}_{32})} = V_1^0(t)\partial_t + \frac{d_1^j c_2^k \epsilon_{jki}}{f_2} \partial_i + d_1^i L_i, \quad V_{2(\mathfrak{C}_{32})} = V_2^0(t)\partial_t + c_2^i\partial_i + f_2 x^i\partial_i. \quad (5.10)$$

Note that type  $\mathfrak{C}_{11}$  and  $\mathfrak{C}_{12}$  can be extended to a triangular  $\star$ -Hopf subalgebra by choosing  $V_1^0(t) = V_2^0(t) = 0$  in each case.

For a better overview we additionally present the the results in table 1, containing all possible two vector field deformations  $\mathfrak{C}_{AB}$  of the Lie algebra of the euclidian group. From this table the notation  $\mathfrak{C}_{AB}$  becomes clear.

Next, we discuss solutions to the third condition (5.4c)  $[V_1^0(t)\partial_t, V_2^0(t)\partial_t] = 0$ . It is obvious that choosing either  $V_1^0(t) = 0$  or  $V_2^0(t) = 0$  and the other one arbitrary is a solution. Additionally, we consider solutions with  $V_1^0(t) \neq 0$  and  $V_2^0(t) \neq 0$ . Therefore there has to be some point  $t_0 \in \mathbb{R}$ , such that without loss of generality  $V_1^0(t)$  is unequal zero in some open region  $U \subseteq \mathbb{R}$  around  $t_0$ . In this region we can perform the diffeomorphism  $t \rightarrow \tilde{t}(t) := \int_{t_0}^t dt' \frac{1}{V_1^0(t')}$  leading to  $\tilde{V}_1^0(\tilde{t}) = 1$ . With this the third condition (5.4c) becomes

$$0 = [V_1^0(t)\partial_t, V_2^0(t)\partial_t] = [\tilde{V}_1^0(\tilde{t})\partial_{\tilde{t}}, \tilde{V}_2^0(\tilde{t})\partial_{\tilde{t}}] = \left(\partial_{\tilde{t}}\tilde{V}_2^0(\tilde{t})\right)\partial_{\tilde{t}}. \quad (5.11)$$

This condition is solved if and only if  $\tilde{V}_2^0(\tilde{t}) = \text{const.}$  for  $t \in U \subseteq \mathbb{R}$ . For the subset of analytical functions  $C^\omega(\mathbb{R}) \subset C^\infty(\mathbb{R})$  we can continue this condition to all  $\mathbb{R}$  and obtain the global relation  $V_2^0(t) = \kappa V_1^0(t)$ , with some constant  $\kappa \in \mathbb{R}$ . For non analytic, but smooth functions, we can not continue these relations to all  $\mathbb{R}$  and therefore only obtain local conditions restricting the functions in the overlap of their supports to be linearly dependent. In particular non analytic functions with disjoint supports fulfill the condition (5.4c) trivially.

After characterizing the possible two vector field deformations of  $\epsilon_3$  we briefly give a method how to obtain twists generated by a larger number of vector fields. For this purpose we use the canonical form of  $\theta$  (4.2).

Assume that we want to obtain deformations with e.g. four vector fields. Then of course all vector fields have to be of the form (5.3). According to the form of  $\theta$  we have two blocks of vector fields  $(a, b) = (1, 2)$  and  $(a, b) = (3, 4)$ , in which the classification described above for two vector fields can be performed. This means that all four vector field twists can be obtained by using two types of two vector field twists. We label the twist by using a tuple of types, e.g.  $(\mathfrak{C}_{11}, \mathfrak{C}_{22})$  means that  $V_1, V_2$  are of type  $\mathfrak{C}_{11}$  and  $V_3, V_4$  of type  $\mathfrak{C}_{22}$ . But this does only assure that  $[V_a, V_b] = 0$  for  $(a, b) \in \{(1, 2), (3, 4)\}$  and we have to demand further restrictions in order to fulfill  $[V_a, V_b] = 0$  for all  $(a, b)$  and that all vector fields give independent contributions to the twist. In particular twists constructed with linearly dependent vector fields can be reduced to a twist constructed by a lower number of vector fields.

This method naturally extends to a larger number of vector fields, until we cannot find anymore independent and mutually commuting vector fields. We will now give two examples for the  $\epsilon_3$  case in order to clarify the construction.

As a first example we construct the four vector field twist  $(\mathfrak{C}_{11}, \mathfrak{C}_{11})$ . In this case all four vector fields commute without imprinting further restrictions. We assume that three of the four vectors  $\mathbf{c}_a$  are linearly independent, such that the fourth one, say  $\mathbf{c}_4$ , can be decomposed into the other ones. If we now choose four linearly independent functions  $V_a^0(t)$  (this means that they are non analytic) leads to a proper four vector field twist.

As a second simple example we construct the four vector field twist  $(\mathfrak{C}_{21}, \mathfrak{C}_{21})$ . In order to have commuting vector fields we obtain the condition  $c_3^i = \frac{f_3}{f_1}c_1^i$ . We therefore have  $V_3 = \frac{f_3}{f_1}V_1$  and the four vector field twist can be reduced to the two vector field twist of type  $\mathfrak{C}_{21}$  with  $\tilde{V}_1 = V_1$  and  $\tilde{V}_2 = V_2 + \frac{f_3}{f_1}V_4$ . This is an example of an improper four vector field twist.



This method can be applied in order to investigate general combinations of two vector field twists, if one requires them. Because this construction is straightforward and we do not require these twists for our discussions, we do not present them here.

At the end we calculate the  $\star$ -commutator of the linear coordinate functions  $x^\mu \in A_\star$  for the various types of models in first order in the deformation parameter  $\lambda$ . It is given by

$$c^{\mu\nu} := [x^\mu \star, x^\nu] := x^\mu \star x^\nu - x^\nu \star x^\mu = i\lambda\theta^{ab}V_a(x^\mu)V_b(x^\nu) + \mathcal{O}(\lambda^2). \quad (5.12)$$

The results are given in appendix B and show that these commutators can be at most quadratic in the spatial coordinates  $x^i$ . Possible applications of these models will be discussed in the outlook, see section 7.

## 6. Application to black holes

In this section we investigate possible deformations of non rotating black holes. We will do this in analogy to the cosmological models and therefore do not have to explain every single step.

The undeformed Lie algebra of the symmetry group  $\mathbb{R} \times \text{SO}(3)$  of a non rotating black hole is generated by the vector fields

$$p^0 = \partial_t \quad , \quad L_i = \epsilon_{ijk}x^j\partial_k, \quad (6.1)$$

given in cartesian coordinates. We choose  $t_i^* = t_i$  for all  $i$  and define  $\mathfrak{g}_\star = \mathfrak{g} = \text{span}(p^0, L_i)$ .

It can be shown that each twist vector field  $V_a$  has to be of the form

$$V_a = (c_a^0(r) + N_a^0 t)\partial_t + d_a^i L_i + f_a(r)x^i\partial_i \quad (6.2)$$

in order to fulfill  $[V_a, \mathfrak{g}] \subseteq \mathfrak{g}$ . Here  $r = \|\mathbf{x}\|$  is the euclidian norm of the spatial position vector.

The next task is to construct the two vector field deformations. Therefore we additionally have to demand  $[V_a, V_b] = 0, \forall_{a,b}$ , leading to the conditions

$$d_a^i d_b^j \epsilon_{ijk} = 0, \quad \forall_k, \quad (6.3a)$$

$$(f_a(r)x^j\partial_j - N_a^0)c_b^0(r) - (f_b(r)x^j\partial_j - N_b^0)c_a^0(r) = 0, \quad (6.3b)$$

$$f_a(r)f_b'(r) - f_a'(r)f_b(r) = 0, \quad (6.3c)$$

where  $f_a'(r)$  means the derivative of  $f_a(r)$ . Note that (6.3c) is a condition similar to (5.4c), and therefore has the same type of solutions. Because of this, the functions  $f_1(r)$  and  $f_2(r)$  have to be parallel in the overlap of their supports. From this we can always eliminate locally one  $f_a(r)$  by a twist conserving map and simplify the investigation of the condition (6.3b). At the end, the local solutions have to be glued together. We choose without loss of generality  $f_1(r) = 0$  for our classification of local solutions.

The solution to (6.3a) is that the  $\mathbf{d}_a$  have to be parallel. We use

$$\mathbf{d}_a = \kappa_a \mathbf{d} \quad (6.4)$$

$\mathfrak{B}_{AB}$	$f_2(r) = 0$	$f_2(r) \neq 0$
$N_1^0 = 0,$ $N_2^0 = 0$	$V_1 = c_1^0(r)\partial_t + \kappa_1 d^i L_i$ $V_2 = c_2^0(r)\partial_t + \kappa_2 d^i L_i$	$V_1 = c_1^0\partial_t + \kappa_1 d^i L_i$ $V_2 = c_2^0(r)\partial_t + \kappa_2 d^i L_i + f_2(r)x^i\partial_i$
$N_1^0 \neq 0,$ $N_2^0 = 0$	$V_1 = (c_1^0(r) + N_1^0 t)\partial_t$ $V_2 = \kappa_2 d^i L_i$	$V_1 = (c_1^0(r) + N_1^0 t)\partial_t + \kappa_1 d^i L_i$ $V_2 = -\frac{1}{N_1^0} f_2(r) r c_1^{0'}(r) \partial_t + \kappa_2 d^i L_i + f_2(r)x^i\partial_i$
$N_1^0 = 0,$ $N_2^0 \neq 0$	$V_1 = \kappa_1 d^i L_i$ $V_2 = (c_2^0(r) + N_2^0 t)\partial_t$	$V_1 = c_1^0(r)\partial_t + \kappa_1 d^i L_i,$ with (6.5) $V_2 = (c_2^0(r) + N_2^0 t)\partial_t + \kappa_2 d^i L_i + f_2(r)x^i\partial_i$

**Table 2:** Two vector field deformations of the black hole symmetry group  $\mathbb{R} \times \text{SO}(3)$ . Note that  $c_1^0(r) = c_1^0$  has to be constant in type  $\mathfrak{B}_{12}$ .

with constants  $\kappa_a \in \mathbb{R}$  and some arbitrary vector  $\mathbf{d} \neq 0$ .

We now classify the solutions to (6.3b) according to  $N_a^0$  and  $f_2(r)$  and label them by  $\mathfrak{B}_{AB}$ . We distinguish between  $f_2(r)$  being the zero function or not. The result is shown in table 2. Other choices of parameters can be mapped by a twist conserving map into these classes. Note that in particular for analytical functions  $f_a(r)$  the twist conserving map transforming  $f_1(r)$  to zero can be performed globally, and with this also the classification of twists given in table 2.

In type  $\mathfrak{B}_{32}$  we still have to solve a differential equation for  $c_1^0(r)$  given by

$$c_1^0(r) = \frac{f_2(r)}{N_2^0} r c_1^{0'}(r), \quad (6.5)$$

for an arbitrary given  $f_2(r)$ . We will not work out the solutions to this differential equation, since type  $\mathfrak{B}_{32}$  is a quite unphysical model, in which the noncommutativity is increasing linear in time due to  $N_2^0 \neq 0$ .

Note that  $\mathfrak{B}_{11}$  can be extended to a triangular  $\star$ -Hopf algebra by choosing  $c_a^0(r) = c_a^0$ , for  $a \in \{1, 2\}$ . In addition,  $\mathfrak{B}_{12}$  is a  $\star$ -Hopf algebra for  $\kappa_1 = \kappa_2 = 0$ .

The  $\star$ -commutators  $c^{\mu\nu} = [x^\mu \star x^\nu]$  of the coordinate functions  $x^\mu \in A_\star$  in order  $\lambda^1$  for these models are given in the appendix B. They can be used in order to construct sensible physical models of a noncommutative black hole.

By using the method explained in the previous section, the two vector field twists can be extended to multiple vector field twists. Since we do not require these twists in our work and their construction is straightforward, we do not present them here.

## 7. Conclusion and outlook

We have discussed symmetry reduction in noncommutative gravity using the formalism of twisted noncommutative differential geometry. Our motivation for these investigations derives from the fact that, for most physical applications of gravity theories, including cosmology, symmetry reduction is required due to the complexity of such models, already in the undeformed case.

In section 3 we have presented a general method for symmetry reduction in twisted gravity theories. As a result we have obtained restrictions on the twist, depending on

the structure of the twisted symmetry group. In particular, we find that deforming the infinitesimal symmetry transformations results in weaker restrictions than deforming the finite transformations and demanding a quantum group structure. In section 4 we have applied this general method to gravity theories twisted by Reshetikhin-Jambor-Sykora twists. These are twists constructed from commuting vector fields. In this case we could give explicit conditions, which have to be fulfilled in order to allow symmetry reduction of a given Lie group.

In sections 5 and 6 we have investigated admissible deformations of FRW and black hole symmetries by a Reshetikhin-Jambor-Sykora twist. In this class we have classified all possible deformations. This lays the foundation for phenomenological studies of noncommutative cosmology and black hole physics based on twisted gravity.

In a forthcoming work [15] we will investigate cosmological implications of twisted FRW models by studying fluctuations of quantum fields living on twisted FRW backgrounds. Quantum fields were already introduced in a twisted framework in [16]. As we see from proposition 4, the noncommutative backgrounds are also invariant under the undeformed action of the classical symmetry. This means that they have the same coordinate representations with respect to the undeformed basis vectors as the commutative fields in Einstein gravity. With this we have a construction principle for noncommutative backgrounds, in their natural basis, by representing the classical fields in the deformed basis. A class of models of particular interest is type  $\mathfrak{C}_{22}$  in section 5 (cf. table 1). These twists break classical translation invariance, but classical rotation invariance can be retained by tuning  $\mathbf{d}_1$  and  $\mathbf{c}_1$  to small values. Furthermore, the global factor  $V_2^0(t)$  in the exponent of the twist can be used in order to tune noncommutativity effects depending on time. Obviously, enforcing a suitable  $V_2^0(t)$  by hand leads to phenomenologically valid models.

Since there is no natural choice of  $V_2^0(t)$ , it is interesting to investigate the dynamics of  $V_2^0(t)$  in a given field configuration and study if it leads to a model consistent with cosmological observations. In this case, the model would be physically attractive. This will also be subject of future work [15]. Dynamical noncommutativity has already been studied in the case of scalar field theories on Minkowski spacetime [17].

In the case of black hole physics, models of particular interest would be  $\mathfrak{B}_{11}$  with functions  $c_a^0(r)$  decreasing sufficiently quickly with  $r$  and  $\mathfrak{B}_{12}$  with  $f_2(r)$  and  $c_2^0(r)$  decreasing sufficiently quickly with  $r$  (cf. table 2). It will again be interesting to investigate the dynamics of these functions on a given field configuration. Note that the type  $\mathfrak{B}_{12}$  with  $\kappa_1 = \kappa_2 = 0$  is invariant under the classical black hole symmetries, and therefore particularly interesting for physical applications. On the other hand, models with nonvanishing  $N_a^0$  are of little physical interest, because the noncommutativity is growing linearly in time, which would be unphysical.

Other avenues for future work are the classification of models on nontrivial topologies (like, e.g.,  $\mathbb{R} \times S_3$  in cosmology), investigating nontrivial embeddings  $t_i^* = t_i^*(t_j)$  and using a wider class of twist elements.

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## A. Proof of proposition 3

In this appendix we show that for Reshetikhin-Jambor-Sykora twists (4.1) the conditions (3.10) necessary for extending the  $\star$ -enveloping subalgebra  $(U\mathfrak{g}_\star, \star)$  to a  $\star$ -Hopf subalgebra are equivalent to the simplified conditions of proposition 3. The plan is as follows: we use (3.10a) and show that it is equivalent to the conditions of proposition 3. In a second step, we show that (3.10b) is automatically satisfied if (3.10a) is fulfilled, and thus does not lead to additional conditions.

We start with (3.10a) and show the identity

$$\Delta_\star(U\mathfrak{g}_\star) \subseteq U\mathfrak{g}_\star \otimes U\mathfrak{g}_\star \Leftrightarrow \Delta_\star(\mathfrak{g}_\star) \subseteq U\mathfrak{g}_\star \otimes U\mathfrak{g}_\star . \quad (\text{A.1})$$

The direction  $\Rightarrow$  is trivial, since  $\mathfrak{g}_\star \subset U\mathfrak{g}_\star$ , and the direction  $\Leftarrow$  can be shown using that  $\Delta_\star$  is a  $\star$ -algebra homomorphism and that  $U\mathfrak{g}_\star$  closes under  $\star$ -multiplication.

Furthermore, using (2.6) we obtain

$$\Delta_\star(\mathfrak{g}_\star) \subseteq U\mathfrak{g}_\star \otimes U\mathfrak{g}_\star \Leftrightarrow X_{\bar{R}^\alpha} \in U\mathfrak{g}_\star , \text{ for all } \alpha \text{ with } \bar{R}_\alpha(\mathfrak{g}_\star) \neq \{0\} . \quad (\text{A.2})$$

Therefore we have to use that all  $\bar{R}^\alpha$  are linearly independent, that  $X$  is a vector space isomorphism [10] and that we have  $\bar{R}_\alpha(\mathfrak{g}_\star) \subseteq \mathfrak{g}_\star$ , due to the minimal axioms (3.6).

Additionally, we can show that  $X_{\bar{R}^\alpha} = \bar{f}^\beta \bar{R}^\alpha \chi S^{-1}(\bar{f}_\beta) = \bar{R}^\alpha$ . This is done by applying the explicit form of the twist (4.1) and using that the  $V_a$  mutually commute.

Next, we show that

$$\bar{R}^\alpha \in U\mathfrak{g}_\star , \text{ for all } \alpha \text{ with } \bar{R}_\alpha(\mathfrak{g}_\star) \neq \{0\} \Leftrightarrow \theta^{ba} V_b \in \mathfrak{g}_\star , \text{ for all } a \text{ with } [V_a, \mathfrak{g}_\star] \neq \{0\} . \quad (\text{A.3})$$

The direction  $\Rightarrow$  is trivial, since the r.h.s. is a special case of the l.h.s. . The direction  $\Leftarrow$  can be shown by using that the  $V_a$  mutually commute and the explicit expression of the  $R$ -matrix (4.3).

Finally, the r.h.s. of (A.3) is equivalent to the condition of proposition 3 by using the canonical form of  $\theta$  (4.2).

Next, we show that (3.10b) is satisfied, if (3.10a) is fulfilled. For this we use that for Reshetikhin-Jambor-Sykora twists we have  $\chi = f^\alpha S(f_\alpha) = 1$ , which leads to the identity

$$S_{\mathcal{F}}(\xi) = \chi S(\xi) \chi^{-1} = S(\xi) = S^{-1}(\xi) = S_{\mathcal{F}}^{-1}(\xi) , \forall \xi \in U\Xi \quad (\text{A.4})$$

for the antipode in the  $\mathcal{F}$ -Hopf algebra. This property translates to the  $\star$ -Hopf algebra, since it is isomorphic to the  $\mathcal{F}$ -Hopf algebra and we obtain the following equivalences of (3.10b)

$$S_\star(U\mathfrak{g}_\star) \subseteq U\mathfrak{g}_\star \Leftrightarrow S_\star(\mathfrak{g}_\star) \subseteq U\mathfrak{g}_\star \Leftrightarrow S_\star^{-1}(\mathfrak{g}_\star) \subseteq U\mathfrak{g}_\star . \quad (\text{A.5})$$

Type	$c^{\mu\nu} := [x^\mu \star x^\nu]$ in $\mathcal{O}(\lambda^1)$
$\mathfrak{C}_{11}$	$c^{0i} = i\lambda(V_1^0(t)c_2^i - V_2^0(t)c_1^i)$ $c^{ij} = i\lambda(c_1^i c_2^j - (i \leftrightarrow j))$
$\mathfrak{C}_{21}$	$c^{0i} = -i\lambda V_2^0(t)(c_1^i + f_1 x^i)$ $c^{ij} = 0$
$\mathfrak{C}_{12}$	$c^{0i} = i\lambda(V_1^0(t)\kappa d_1^i - V_2(t)(c_1^i + d_1^k \epsilon_{kli} x^l))$ $c^{ij} = i\lambda\kappa((c_1^i + d_1^k \epsilon_{kli} x^l) d_1^j - (i \leftrightarrow j))$
$\mathfrak{C}_{22}$	$c^{0i} = -i\lambda V_2^0(t)(c_1^i + d_1^j \epsilon_{jki} x^k + f_1 x^i)$ $c^{ij} = 0$
$\mathfrak{C}_{32}$	$c^{0i} = i\lambda(V_1^0(t)(c_2^i + f_2 x^i) - V_2^0(t)(\frac{1}{f_2} d_1^j c_2^k \epsilon_{jki} + d_1^j \epsilon_{jki} x^k))$ $c^{ij} = i\lambda((\frac{1}{f_2} d_1^k c_2^l \epsilon_{kli} + d_1^k \epsilon_{kli} x^l)(c_2^j + f_2 x^j) - (i \leftrightarrow j))$

**Table 3:**  $\star$ -commutators in the cosmological models  $\mathfrak{C}_{AB}$ .

Type	$c^{\mu\nu} := [x^\mu \star x^\nu]$ in $\mathcal{O}(\lambda^1)$
$\mathfrak{B}_{11}$	$c^{0i} = i\lambda(c_1^0(r)\kappa_2 - c_2^0(r)\kappa_1)d^j \epsilon_{jki} x^k$ $c^{ij} = 0$
$\mathfrak{B}_{21}$	$c^{0i} = i\lambda(c_1^0(r) + N_1^0 t)d^j \epsilon_{jki} x^k$ $c^{ij} = 0$
$\mathfrak{B}_{12}$	$c^{0i} = i\lambda(c_1^0(\kappa_2 d^j \epsilon_{jki} x^k + f_2(r)x^i) - \kappa_1 c_2^0(r)d^j \epsilon_{jki} x^k)$ $c^{ij} = i\lambda(\kappa_1 d^k \epsilon_{kli} x^l (\kappa_2 d^m \epsilon_{mnj} x^n + f_2(r)x^j) - (i \leftrightarrow j))$
$\mathfrak{B}_{22}$	$c^{0i} = i\lambda((c_1^0(r) + N_1^0 t)(\kappa_2 d^j \epsilon_{jki} x^k + f_2(r)x^i) + \frac{1}{N_1^0} f_2(r) r c_1^{0l}(r) \kappa_1 d^j \epsilon_{jki} x^k)$ $c^{ij} = i\lambda(\kappa_1 d^k \epsilon_{kli} x^l (\kappa_2 d^m \epsilon_{mnj} x^n + f_2(r)x^j) - (i \leftrightarrow j))$
$\mathfrak{B}_{32}$	$c^{0i} = i\lambda(c_1^0(r)(\kappa_2 d^j \epsilon_{jki} x^k + f_2(r)x^i) - (c_2^0(r) + N_2^0 t)\kappa_1 d^j \epsilon_{jki} x^k)$ $c^{ij} = i\lambda(\kappa_1 d^k \epsilon_{kli} x^l (\kappa_2 d^m \epsilon_{mnj} x^n + f_2(r)x^j) - (i \leftrightarrow j))$

**Table 4:**  $\star$ -commutators in the black hole models  $\mathfrak{B}_{AB}$ .

For the first equivalence we had to use that  $S_\star$  is a  $\star$ -anti homomorphism.

Using the r.h.s. of (A.3), which is equivalent to (3.10a), and the definition of  $S_\star^{-1}$  (2.6), we obtain

$$S_\star^{-1}(\mathfrak{g}_\star) = - \sum_{n=0}^{\infty} \frac{(-i\lambda)^n}{n!} \theta^{a_1 b_1} \dots \theta^{a_n b_n} [V_{a_1}, \dots, [V_{a_n}, \mathfrak{g}_\star] \dots] V_{b_1} \dots V_{b_n} \in U\mathfrak{g}_\star, \quad (\text{A.6})$$

where we have used  $\xi \star V_a = \xi V_b$  for all  $\xi \in U\Xi_\star$ , since the action of the twist on  $V_a$  is trivial.

## B. $\star$ -commutators of the coordinate functions in FRW and black hole models

In tables 3 and 4, we list the  $\star$ -commutators among the linear coordinate functions to order  $\lambda^1$  in the FRW and black hole models. In these expressions,  $(i \leftrightarrow j)$  denotes the same term with  $i$  and  $j$  interchanged.

## References

- [1] S. Doplicher, K. Fredenhagen and J.E. Roberts, *Space-time quantization induced by classical gravity*, *Phys. Lett. B* **331** (1994) 39; *The quantum structure of space-time at the Planck scale and quantum fields*, *Commun. Math. Phys.* **172** (1995) 187 [[hep-th/0303037](#)].
- [2] R.J. Szabo, *Quantum field theory on noncommutative spaces*, *Phys. Rept.* **378** (2003) 207 [[hep-th/0109162](#)]; *Symmetry, gravity and noncommutativity*, *Class. and Quant. Grav.* **23** (2006) R199 [[hep-th/0606233](#)].
- [3] F. Muller-Hoissen, *Noncommutative geometries and gravity*, *AIP Conf. Proc.* **977** (2008) 12 [[arXiv:0710.4418](#)].
- [4] X. Calmet, B. Jurčo, P. Schupp, J. Wess and M. Wohlgenannt, *The standard model on non-commutative space-time*, *Eur. Phys. J. C* **23** (2002) 363 [[hep-ph/0111115](#)];  
B. Melic, K. Pasek-Kumericki, J. Trampetic, P. Schupp and M. Wohlgenannt, *The standard model on non-commutative space-time: electroweak currents and Higgs sector*, *Eur. Phys. J. C* **42** (2005) 483 [[hep-ph/0502249](#)]; *The standard model on non-commutative space-time: strong interactions included*, *Eur. Phys. J. C* **42** (2005) 499 [[hep-ph/0503064](#)].
- [5] W. Behr et al., *The  $Z \rightarrow \gamma\gamma$ ,  $g g$  decays in the noncommutative standard model*, *Eur. Phys. J. C* **29** (2003) 441 [[hep-ph/0202121](#)];  
G. Duplancic, P. Schupp and J. Trampetic, *Comment on triple gauge boson interactions in the non-commutative electroweak sector*, *Eur. Phys. J. C* **32** (2003) 141 [[hep-ph/0309138](#)];  
P. Schupp, J. Trampetic, J. Wess and G. Raffelt, *The photon neutrino interaction in non-commutative gauge field theory and astrophysical bounds*, *Eur. Phys. J. C* **36** (2004) 405 [[hep-ph/0212292](#)];  
P. Minkowski, P. Schupp and J. Trampetic, *Neutrino dipole moments and charge radii in non-commutative space-time*, *Eur. Phys. J. C* **37** (2004) 123 [[hep-th/0302175](#)];  
B. Melic, K. Pasek-Kumericki and J. Trampetic, *Quarkonia decays into two photons induced by the space-time non-commutativity*, *Phys. Rev. D* **72** (2005) 054004 [[hep-ph/0503133](#)];  
 *$K \rightarrow \pi\gamma$  decay and space-time noncommutativity*, *Phys. Rev. D* **72** (2005) 057502 [[hep-ph/0507231](#)];  
M. Burić, D. Latas, V. Radovanović and J. Trampetic, *Nonzero  $Z \rightarrow \gamma\gamma$  decays in the renormalizable gauge sector of the noncommutative standard model*, *Phys. Rev. D* **75** (2007) 097701;  
T. Ohl and J. Reuter, *Testing the noncommutative standard model at a future photon collider*, *Phys. Rev. D* **70** (2004) 076007 [[hep-ph/0406098](#)];  
A. Alboteanu, T. Ohl and R. Ruckl, *Probing the noncommutative standard model at hadron colliders*, *Phys. Rev. D* **74** (2006) 096004 [[hep-ph/0608155](#)]; *The noncommutative standard model at the ILC*, *eConf C0705302* (2007) TEV05 [*Acta Phys. Polon.* **B38** (2007) 3647] [[arXiv:0709.2359](#)]; *The noncommutative standard model at  $O(\theta^2)$* , *Phys. Rev. D* **76** (2007) 105018 [[arXiv:0707.3595](#)].
- [6] I. Hinchliffe, N. Kersting and Y.L. Ma, *Review of the phenomenology of noncommutative geometry*, *Int. J. Mod. Phys. A* **19** (2004) 179 [[hep-ph/0205040](#)].
- [7] C.-S. Chu, B.R. Greene and G. Shiu, *Remarks on inflation and noncommutative geometry*, *Mod. Phys. Lett. A* **16** (2001) 2231 [[hep-th/0011241](#)];  
F. Lizzi, G. Mangano, G. Miele and M. Peloso, *Cosmological perturbations and short distance physics from noncommutative geometry*, *JHEP* **06** (2002) 049 [[hep-th/0203099](#)];

- R. Brandenberger and P.-M. Ho, *Noncommutative spacetime, stringy spacetime uncertainty principle and density fluctuations*, *Phys. Rev. D* **66** (2002) 023517 [*AAPPs Bull.* **12N1** (2002) 10] [[hep-th/0203119](#)];
- Q.-G. Huang and M. Li, *CMB power spectrum from noncommutative spacetime*, *JHEP* **06** (2003) 014 [[hep-th/0304203](#)]; *Noncommutative inflation and the CMB multipoles*, *JCAP* **11** (2003) 001 [[astro-ph/0308458](#)];
- S. Tsujikawa, R. Maartens and R. Brandenberger, *Non-commutative inflation and the CMB*, *Phys. Lett. B* **574** (2003) 141 [[astro-ph/0308169](#)];
- A.H. Fatollahi and M. Hajirahimi, *Noncommutative black-body radiation: implications on cosmic microwave background*, *Europhys. Lett.* **75** (2006) 542 [[astro-ph/0607257](#)];
- E. Akofor, A.P. Balachandran, S.G. Jo, A. Joseph and B.A. Qureshi, *Direction-dependent CMB power spectrum and statistical anisotropy from noncommutative geometry*, *JHEP* **05** (2008) 092 [[arXiv:0710.5897](#)];
- E. Akofor, A.P. Balachandran, A. Joseph, L. Pekowsky and B.A. Qureshi, *Constraints from CMB on spacetime noncommutativity and causality violation*, [arXiv:0806.2458](#);
- S. Fabi, B. Harms and A. Stern, *Noncommutative corrections to the Robertson-Walker metric*, *Phys. Rev. D* **78** (2008) 065037 [[arXiv:0808.0943](#)].
- [8] P. Nicolini, *Noncommutative black holes, the final appeal to quantum gravity: a review*, [arXiv:0807.1939](#).
- [9] P. Aschieri et al., *A gravity theory on noncommutative spaces*, *Class. and Quant. Grav.* **22** (2005) 3511 [[hep-th/0504183](#)].
- [10] P. Aschieri, M. Dimitrijević, F. Meyer and J. Wess, *Noncommutative geometry and gravity*, *Class. and Quant. Grav.* **23** (2006) 1883 [[hep-th/0510059](#)].
- [11] P. Aschieri, *Noncommutative symmetries and gravity*, *J. Phys. Conf. Ser.* **53** (2006) 799 [[hep-th/0608172](#)].
- [12] L. Álvarez-Gaumé, F. Meyer and M.A. Vazquez-Mozo, *Comments on noncommutative gravity*, *Nucl. Phys. B* **753** (2006) 92 [[hep-th/0605113](#)].
- [13] N. Reshetikhin, *Multiparameter quantum groups and twisted quasitriangular Hopf algebras*, *Lett. Math. Phys.* **20** (1990) 331.
- [14] C. Jambor and A. Sykora, *Realization of algebras with the help of \*-products*, [hep-th/0405268](#).
- [15] T. Ohl and A. Schenkel, *Cosmological and black hole solutions in twisted noncommutative gravity*, to be published; *Classical and quantum field fluctuations on noncommutative backgrounds*, in preparation.
- [16] P. Aschieri, F. Lizzi and P. Vitale, *Twisting all the way: from classical mechanics to quantum fields*, *Phys. Rev. D* **77** (2008) 025037 [[arXiv:0708.3002](#)].
- [17] P. Aschieri, L. Castellani and M. Dimitrijević, *Dynamical noncommutativity and Nöther theorem in twisted  $\phi^{*4}$  theory*, *Lett. Math. Phys.* **85** (2008) 39 [[arXiv:0803.4325](#)].